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MODELLING ICE FLOW IN VARIOUS GLACIER ZONES†

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A simplified model of plane isothermal steady ice flow in the land-based part, the shelf and the ice-sheet-ice-shelf transition zone of a glacier which interacts with the sea is constructed using Glen's flow law and perturbation methods. Boundary conditions on the joining line are obtained for the land-based part of the glacier. The dynamics of the ice-divide domain of the glacier is analysed for the case of axially symmetric, steady, isothermal ice flow. © 2001 Elsevier Science Ltd. All rights reserved.

Laboratory investigations [1] have shown that the motion of ice can be described using a model of the flow of an incompressible fluid with an exponential rheological law and a creep coefficient close to three (Glen's law). In the case of glaciers, the Reynolds number is of the order of 10^{-11} and the Stokes approximation, which describes inertialess flow, is therefore used. It has been shown [2] that, in the case of many glaciers, the effect of temperature can be taken into account by the choice of its characteristic value in the layer close to the base. On account of this, we will henceforth confine ourselves to considering two-dimensional isothermal flow. We shall also assume that there is no ablation and slippage of ice on the bed [3] (which is assumed to be solid). The viscosity of water is neglected compared with the viscosity of ice (in the floating ice region) and the high-frequency components of the function for the bed profile are also neglected (the characteristic slopes of the bed and the upper surface of the land-based glacier are assumed to be equal).

When investigating marine glaciers (Fig. 1), four parts are distinguished the land-based glacier, the shelf glacier (hereinafter, simply called the shelf, that is, the floating part of the glacier), the glacier – shelf transition zone and the ice-divide domain (the ice divide means the centre of the ice spreading). The majority of gaps are bored in the last zone since the horizontal velocities are minimal in it, which facilitates the dating of an extracted core sample. When modelling plane flow it is frequently assumed that the glacier is symmetrical about a stationary plane of symmetry which, in the present case, also determines the ice divide. In the case of axisymmetric flow, the ice divide coincides with the axis of symmetry.

Steady flow of a land-based glacier is possible when there is a zero mass balance across its upper surface and the drain into the ice shelf. A steady-state condition of the ice shelf is ensured by zero balance in the inflow of ice from the land-based glacier across the surface and outflow as a consequence of the icebergs chipping off.

The land-based part of a glacier and the ice shelf are characterized by a low ratio of their characteristic thicknesses to their lengths (for example, the length, the thickness and the characteristic time of change of the land-based parts of Antarctic glaciers are estimated to be 10^3 km, 3 km and 10^4 years respectively). This enables one to simplify the problem using the thin-layer approximation and to obtain models describing the shear flow in the land-based part of a glacier [4] (when no-slip conditions are set up on the bed) and the expansion flow in the ice shelf [5]. However, to solve these problems, a knowledge is required of the conditions of the matching of the ice thickness along the joining line (where the ice begins to rise to the surface), which are boundary conditions on the ice thickness function in the case of problems of land-based ice and an ice shelf. In order to determine these conditions, it is necessary to investigate the glacier-ice-shelf transition zone in which the thin-layer approximation is unsuitable and where the shear and normal stresses are of the same order of magnitude, since a transition occurs from shear flow in the land-based part of the glacier to expansion flow in the ice shelf.

As a rule, the condition of hydrostatic equilibrium of the ice and a mass balance condition, which are simplified in the thin-layer approximation, are set up on the joining line. Generally speaking, these condition are unfounded. These relations have been modified in [7] by using an unknown parameter

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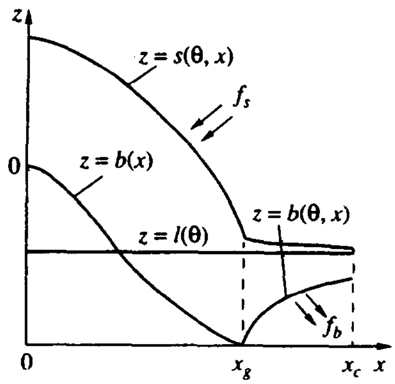

Fig. 1

which characterizes the deviation of the normal deviator stresses, obtained in the thin-layer approximation, from their true value. However, the algorithm for finding the value of this parameter has not been defined. At the same time, as numerical experiments in [8] showed, the position of the joining line is sensitive to a change in the parameters occurring in the boundary condition on it. The problem therefore arises of finding the unknown fields in the transition zone and of obtaining rigorously substantiated boundary conditions on the joining line. An analysis of this problem for ice of a constant viscosity has been carried out in [9] but, since the viscosity of ice is non-linear, the applicability of these results is limited and it is necessary to consider the general case of a power-function rheology.

A mathematical model of a land-based glacier, obtained in the thin-layer approximation, has been found, neglecting the normal deviator stresses. In this case, for a power-function rheological law, the viscosity is proportional to the negative power of the shear stress. This model is therefore unsuitable in the ice-divide domain, since, in the plane (or axis) of symmetry of the glacier, the shear stresses are equal to zero, and this leads to an infinite viscosity. This characteristic determines the infinite curvature of the upper surface at the ice divide [10]. It has been shown in [11] that, in the case of plane ice flow in the ice domain, the shear and normal stresses are of the same order of magnitude, and it is therefore necessary to solve the complete Stokes system. A simplified model of the dynamics of the ice-divide domain in the case of plane ice flow has been constructed [12] using perturbation methods. However, many ice domes are almost axisymmetric in shape and a mathematical analysis of the corresponding problem is necessary. Earlier, this problem had only been solved numerically in [13] where steady flow was considered.

The foregoing discussion determines the aim of this paper, that is, to construct a well-founded simplified mathematical model, first, of an ice sheet interacting with an ocean when there is a plane, steady flow of ice (Section 2), which takes account of the features of its basic zones: the land-based glacier, the ice shelf and the glacier-shelf transition zone and, second, of the ice-divide of the land-based glacier in the case of axisymmetric, steady ice flow (Section 3).

1. BASIC EQUATIONS

Consider a plane or axisymmetric flow of a glacier which is running into an ocean (Fig. 1). We place the origin of a Cartesian cylindrical system of coordinates at the bed level on the ice-divide. The z axis is directed upwards and belongs to the plane of symmetry in the case of plane flow or coincides with the axis of symmetry in the case of axisymmetric flow and the x axis is directed horizontally along the flow. The domain $0 < x < x_g$, where x_g is the position of the joining line, will determine the land-based part of the glacier and the domain $x_g < x < x_c$, where x_c is the line of cleavage of the ice shelf, will determine the floating part.

Notation: θ is the time, $s(\theta, x)$ is the profile of the glacier upper surface, $b(\theta, x)$ is the profile of the glacier lower surface, which coincides with the domain $x < x_g$ with a rise of the fixed, rigid bed, p is the pressure, u is the horizontal velocity of the particles along the x axis, w is the vertical velocity of the particles, τ_1, τ_2, τ_3, τ_4 are the shear stress and the normal deviator stresses along the x axis, the z axis and (in the case of axisymmetric flow) the φ axis respectively, μ^* is the coefficient of the effective viscosity, η is the rheological constant in the rheological law [1]

$$\mu^* = 2^{1/\alpha-1}\eta^{1/\alpha}e_{(2)}^{(1-\alpha)/(2\alpha)}, \qquad e_{(2)} = \frac{1}{2}\left[u_x^2 + w_z^2 + \frac{1}{2}(u_z + w_x)^2 + \left(\frac{nu}{x}\right)^2\right]$$

where $e_{(2)}$ is the second invariant of the deformation note tensor, $\alpha \geqslant 1$ is the creep coefficient, f_s, $(-f_b)$ are the rates of accumulation of ice at the upper and lower surfaces, respectively (it is assumed that there is no ablation in the bed: $f_b = 0, x < x_g$), g is the acceleration due to gravity, ρ_i and ρ_w are the densities of ice and water, l is the sea level, $k = \rho_i/\rho_w$, $\delta = 1 - k$ is the normalized difference between the densities of water and ice, ε is the characteristic slope of the glacier surface, and $n = 0$ in the case of a plane flow and $n = 1$ in the case of axisymmetric flow. Henceforth, in the various expressions, t_i, T_i and e_i, E_i, apart from multipliers, have the physical meaning of stresses and deformation rates, and the subscripts, which correspond to independent variables, denote partial derivatives.

Unsteady ice flow in the selected system of coordinates, when account is taken of the incompressibility of the ice, is described by the Stokes equations

$$\begin{aligned} &-(x^n p)_x + (x^n\tau_2)_x + (x^n\tau_1)_z = n(-p+\tau_4) \\ &-(x^n p)_z + (x^n\tau_1)_x + (x^n\tau_3)_z = x^n\rho_i g \qquad (1.1) \\ &\tau_2 + \tau_3 + n\tau_4 = 0, \quad (\tau_1,\tau_2,\tau_3,\tau_4) = \mu^*\left(u_z + w_x, 2u_x, 2w_z, \frac{2u}{x}\right) \\ &\mu^* = \eta^{1/\alpha}\left[2u_x^2 + 2w_z^2 + (u_z + w_x)^2 + 2\left(\frac{nu}{x}\right)^2\right]^{(1-\alpha)/(2\alpha)}, \quad 0 < x < x_c, \quad b < z < s \end{aligned}$$

where the expression for the effective viscosity follows from the definition of the rheological law presented above.

The boundary conditions depend on the process being considered. The stresses on the upper surface are equal to zero and the law of conservation of mass holds. A no-slip condition is used on the bed [3]. On the lower surface of the ice, in the domain where it floats, the shear stress is equal to zero (when the viscosity of water is neglected), the normal stress is the hydrostatic pressure of the water and the law of conservation of mass holds. At the ice divide, the flow is assumed to be symmetric and the horizontal velocity is equal to zero. Along the line of cleavage of the ice shelf, the position of which is assumed given, as will be obvious in the solution, it is sufficient to determine the magnitude of the normal force [14], the value of which in water is known. Hence, the boundary conditions have the form

$$z = s: \quad s_x(-p+\tau_2) = \tau_1, \quad s_x\tau_1 = -p + \tau_3, \quad s_\theta + us_x = w + f_s \qquad (1.2)$$

$$z = b, \ x > x_g: \quad b_x(p-\tau_2) = b_x(l-b)\rho_w g - \tau_1, \quad b_x\tau_1 + p - \tau_3 = (l-b)\rho_w g$$

$$b_\theta + ub_x = w + f_b \qquad (1.3)$$

$$z = b, \quad x < x_g: \quad u = w = 0; \quad x = 0: \quad u = w_x = 0 \qquad (1.4)$$

$$x = x_c: \quad \int_b^s (-p+\tau_2)dx = -\frac{1}{2}\rho_w g(l-b)^2 \qquad (1.5)$$

Far from the ocean, the ice flow is described in the thin-layer approximation where shear stresses play a leading role and the ratio of the characteristic thickness of the ice to the length of the land-based glacier ε is small.

Henceforth, we shall omit the initial conditions for the ice surface.

In order to reduce relations (1.1)–(1.5) to dimensionless form, we use the scales found using similarity theory [4] (these scales are denoted below by square brackets)

$$\begin{aligned} &[p] = \rho_i g[z], \quad [w] = [f_s] = [f_b], \quad [u] = [x][f_s][z]^{-1}, \quad [s] = [z] = [b] = [l] = \varepsilon[x] \\ &\varepsilon = ([f_s]\eta(\rho_i g)^{-\alpha}[x]^{-\alpha-1})^{1/(2\alpha+2)}, \quad [\mu^*] = \eta^{1/\alpha}([x][f_s][z]^{-2})^{1/\alpha-1}, \quad [\theta] = [z][f_s]^{-1} \qquad (1.6) \end{aligned}$$

The length scale of the glacier $[x]$ and the rate of ice accumulation on the surface $[f_{s,b}]$ are determined from observations. Physically, this relation between ε, f_s and the remaining parameters ensures equality of the scales of the specific force caused by the vertical change in the shear stress and the pressure drop in the horizontal direction.

We will consider the case when the amplitude of the high-frequency components of the bed profile functions is small compared with the ice thickness and we shall therefore neglect them and actually assume that $b_s = O(\varepsilon), x < x_g$. We will also assume that, at the point $x = 0$, the bed profile is described by a smooth curve, and, therefore, as a consequence of symmetry, $b_x(0) = 0$.

2. CONSTRUCTION OF A MODEL OF A MARINE GLACIER FOR PLANE STEADY ICE FLOW

We will assume that steady flow of the glacier exists and change to dimensionless coordinates, which are denoted by capital letters. Since the main problem of this section is associated with the modelling of the transition zone, for convenience in writing the formulae, we will change to a system of coordinates at a point of the bed on the joining line which, in this case, will be defined as $X = 0$. It is natural that, here, the ice divide position is given by the expression $X = -X_g$.

When $n = 0$, system (1.1)–(1.5), can be reduced to the problem of determining the stream function $\Psi(U = \Psi_Z, W = -\Psi_X)$ and the unknown upper surface of the ice S and the lower surface of the ice shelf $(X > 0)$ B:

$$T_{1ZZ} + 2\varepsilon^2 T_{2XZ} - \varepsilon^2 T_{1XX} = 0, \quad -X_g < X < X_c, \quad B < Z < S \tag{2.1}$$

$$Z = S: \quad E_1(1 - \varepsilon^2 S_X^2) = 2\varepsilon^2 S_x E_2, \quad \Psi = \int_{-X_g}^{X} F_s dX \tag{2.2}$$

$$Z = B, \quad X < 0: \quad \Psi = \Psi_Z = 0 \tag{2.3}$$

$$Z = B, \quad X > 0: \quad E_1(1 - \varepsilon^2 B_X^2) = 2\varepsilon^2 B_X E_2, \quad \Psi = \int_0^X F_b dX \tag{2.4}$$

$$X = -X_g: \quad \Psi_Z = \Psi_{XX} = 0 \tag{2.5}$$

where

$$(T_1, T_2) = \mu(E_1, E_2), \quad E_1 = \Psi_{ZZ} - \varepsilon^2 \Psi_{XX}, \quad E_2 = 2\Psi_{XZ}, \quad \mu = [\varepsilon^2 E_2^2 + E_1^2]^{(1-\alpha)/(2\alpha)}$$

The equations of the ice surfaces have the form

$$H^2 - \frac{1}{k}(L - B)^2 = 2\varepsilon^2 G, \quad B - S + kH = k\varepsilon^2 T_x \text{ when } X > 0, \quad B(+0) = 0 \tag{2.6}$$

$$H^2 - \frac{1}{k} L^2 = 2\varepsilon^2 G + 2\int_X^0 (HB_X + [IB_X + (1 + \varepsilon^2 B_X^2)\mu\Psi_{ZZ}]_{Z=B})dX \text{ when } X < 0 \tag{2.7}$$

where, for brevity, the following notation has been used for the integral forces

$$G = \int_B^S [2T_2 + J_X]dZ, \quad I = -\varepsilon^2[T_2 + J_X], \quad J = \int_Z^S T_1 dZ, \quad T = J|_{Z=B}, \quad H = S - B \tag{2.8}$$

Equations (2.1)–(2.5) are obtained from problem (1.1)–(1.5) by eliminating the function p. The first equality of (2.6) and equality (2.7) are the result of successive integration of the first equation of (1.1) with respect to z from b to s and, also, with respect to x from x to x_c and from x to x_g respectively. The second relation of (2.6) is obtained from the second equation of (1.1) by integrating with respect to z from b to s.

After reduction to dimensionless variables, the resulting equations take the form (2.1)–(2.7).

From the physical point of view, the first equality of (2.6) and equality (2.7) are the equations for the balance of the forces for the volumes of ice defined by the domains (X, X_c) and $(X, 0)$, and the second equation of (2.6) is for the cross-section X = const. Analogous equations for determining the ice surfaces will subsequently be obtained from (2.6) and (2.7) by a transformation of the coordinates and they therefore have the same physical meaning.

The use of equations in integral form for determining the ice surface is due to the fact that stresses can be generated in them (this is clear, for example, from relations (2.9), which are presented below) which makes the analysis difficult.

All terrestrial glaciers are characterized by a characteristic slope of the glacier surface $\varepsilon \simeq 10^{-2}$–$10^{-3}$. This parameter reflects the conditions of their existence.

Model of a land-based glacier. We will seek outer expansions $\overline{\Psi}$, $\overline{S}$ of the stream function Ψ and the height of the surface S in the domain of a land-based glacier in the form

$$\overline{\Psi} = \overline{\Psi}_0 + O(\varepsilon^2), \quad \overline{S} = \overline{S}_0 + O(\varepsilon^2)$$

From relations (2.1)–(2.8), we then find

$$\overline{\Psi}_0 = \frac{Q}{\alpha+1}\left[(\alpha+2)\frac{Z-B}{\overline{H}_0} - 1 + \frac{(\overline{S}_0 - Z)^{\alpha+2}}{\overline{H}_0^{\alpha+2}}\right]$$

$$\overline{S}_{0x} = -\left[(\alpha+2)\frac{Q}{\overline{H}_0^{\alpha+2}}\right]^{1/\alpha}; \quad Q = \int_{-X_g}^{X} F_s dX^-, \overline{H}_0 = \overline{S}_0 - B \tag{2.9}$$

in a similar manner to that described earlier in [4]. Here, Q is the horizontal mass flow.

Equation (2.9) for the upper surface does not include the stream function and serves to determine the profile of the ice surface far from the joining line. This problem can be solved numerically, since it is difficult to solve analytically when the bed has an arbitrary profile. When the bed is flat and horizontal, and the rate of accumulation is constant, the analytical solution of this problem is well-known [15]. Since (2.9) is a first-order equation, in order to obtain a unique solution it is necessary to find the height of the surface of the outer expansion along the joining line $\overline{S}_0(0)$ and the quantity X_g, which is also unknown. These parameters must be determined by matching the profiles of the upper surfaces for the different glacial zones.

The outer expansion which has been constructed is characterized by an infinite curvature of the upper surface at the point $X = -X_g$ close to which it is unsuitable [10, 11]. This singularity can be taken into account by constructing an inner expansion in the ice-divide domain [12].

Model of the ice flow in the shelf. In order to use the theory of perturbation methods to model the ice flow in the shelf, it is necessary to change to a system of coordinates normalized to scales which are characteristic of the given flow domain. The characteristic length of the shelf and of the land-based glacier are taken to be equal. We now introduce the unknown scale for the shelf thickness $[z_s]$ and the quantity $\Delta_s = [z_s]/[z] \leqslant 1$, and change to the variables

$$(\tilde{X}, \tilde{\Psi}) = (X, \Psi), \quad \tilde{V} = \frac{V}{\Delta}, \quad V = (Z, S, B, L), \quad \varepsilon_s = \frac{[z_s]}{[x]} \leqslant \varepsilon$$

In these new variables, the equation and the boundary condition for the stream function look similar to (2.1)–(2.5) if, in these equations, the symbols (X, Z, ε) are replaced by $(\tilde{X}, \tilde{Z}, \varepsilon_s)$, respectively.

The equations of the ice surfaces (2.6) give

$$\tilde{H}^2 = \frac{2^{2+1/\alpha}}{\delta}\left(\frac{\varepsilon}{\Delta_s}\right)^{1+1/\alpha} \int_{\tilde{B}}^{\tilde{S}} \tilde{\Psi}_{\tilde{X}\tilde{Z}}^{1/\alpha} d\tilde{Z} + O(\varepsilon_s^2), \quad \tilde{H} = \tilde{S} - \tilde{B} \tag{2.10}$$

Equating the factor in front of the bracket to unity, the characteristic thickness of the shelf can be found. In the case of an arbitrary length scale for the shelf $[x_s] \leqslant [x]$, the formula is obtained in a similar manner and has the form

$$[z_s] = (\eta 2^{2\alpha+1}(\rho_i g\delta)^{-\alpha}[x][f_s][x_s]^{-1})^{1/(\alpha+1)} \tag{2.11}$$

Expanding the required functions in asymptotic series in powers of ε_s^2 and taking account of the fact that $\tilde{\Psi}_{\tilde{Z}\tilde{Z}} = O(\varepsilon_s^2)$, which follows from Eq. (2.1), we find the leading terms of the outer expansion

$$\tilde{\Psi}_0 = Q\frac{\tilde{Z}-\tilde{L}}{\tilde{H}_0} + kQ + \int_0^{\tilde{X}} F_b dX, \quad \tilde{S}_0 = \tilde{L} + \delta\tilde{H}_0, \quad \tilde{B}_0 = \tilde{L} - k\tilde{H}_0 \tag{2.12}$$

$$\tilde{H}_0 = Q\left(A + (\alpha+1)\int_0^{\tilde{X}} Q^\alpha dX\right)^{-1/(\alpha+1)}; \quad Q = \int_{-\tilde{X}_g}^{\tilde{X}} F dX, \quad F = F_s - F_b, \quad A = \text{const}$$

The resulting analytical solution, which describes expansion flow (the predominance of normal stresses over shear stresses) is identical with that found previously (see [5] and the references in it). However, the relation between the length and thickness scales of the shelf (2.11) has been obtained for the first time. This solution describes a "slab" flow, which is characterized by the fact that there is no change in the horizontal velocity with respect to the vertical velocity. The parameter A is determined from matching the expansions for the different zones and determines the ice thickness of the outer expansion along the joining line. Obviously, the occurrence of an inflow of ice from a land-based glacier corresponds to the finite thickness of the ice along the joining line, and, if $0 < A < \infty$, steady flow of the shelf glacier occurs.

It is impossible to match the outer expansions for the stream function in the domains of a land-based glacier and the shelf without introducing a boundary layer in the neighbourhood of the point $X = 0$ and constructing an inner expansion which will also determine the transition zone.

Model of the transition zone. Let us find the spatial scales of the transition zone. Taking account of the no-slip conditions on the bed, it is natural to select the scale for the ice thickness as the vertical scale. We will now determine the horizontal scale. Note that normal stresses predominate in the shelf domain while shear stresses predominate in the land-based part of the glacier. Hence, in the transition zone, where a change in the flow conditions occurs, the normal and shear stresses are of the same order of magnitude. In the land-based part of the glacier, only two terms are decisive in the first equation of (1.1): the pressure gradient and the specific force due to shear deformations. In the shelf, all of the forces (the terms in the first equation of (1.1)) are equivalent. Hence, in this equation, which refers to the transition zone, all the terms must be of the same order of magnitude, which is only possible if the horizontal and vertical scales are identical. The horizontal spatial scale of the transition zone is therefore equal to the scale for the ice thickness along the joining line and, in particular, the ratio between them is independent of ε and δ. Moreover, since, in the transition zone, the stress scales, as well as the spatial scales, are the same, it is impossible to match the surfaces of the ice and obtain the boundary conditions on the joining line solely from the balance equations. This can be shown by integrating the equations of motion over the domain of the transition zone (see relation (2.15) below, which cannot be simplified to a form which does not contain the value of the stream function in the transition zone).

We will denote the ice thickness along the joining line by $\Delta = H(0)$ and introduce the inner expansion variables

$$\xi = \frac{X}{\varepsilon\Delta}, \quad (y, \chi, \vartheta, \gamma, h) = \frac{1}{\Delta}(Z, S, B, L, H), \quad \psi = \frac{\Psi}{Q_0}; \quad Q_0 = \int_{-X_g}^{0} F_s dX$$

where Q_0 is the (unknown) mass flow along the joining line and we rewrite relations (2.1)–(2.8) in the new variables

$$t_{1yy} + 2t_{2\xi y} - t_{1\xi\xi} = 0, \quad -\infty < \xi < +\infty, \quad \vartheta < y < \chi$$
$$y = \chi: \quad e_1(1-\chi_\xi^2) = 2\chi_\xi e_2, \quad \psi = 1 + O(\varepsilon); \quad y = \vartheta, \quad \xi < 0: \quad \psi = \psi_y = 0 \tag{2.13}$$
$$y = \vartheta, \quad \xi > 0: \quad e_1(1-\vartheta_\xi^2) = 2\vartheta_\xi e_2, \quad \psi = O(\varepsilon)$$

where

$$e_1 = \psi_{yy} - \psi_{\xi\xi}, \quad e_2 = 2\psi_{\xi y}, \quad (t_1, t_2) = \mu(e_1, e_2), \quad \mu = (e_1^2 + e_2^2)^{(1-\alpha)/(2\alpha)}$$

The equations for the height of the surfaces (balancing of the forces) have the form

$$\beta h^2 + 2kht_\xi - k\delta\beta^{-1}t_\xi^2 = 2\zeta, \quad \vartheta - \gamma + kh = k\delta\beta^{-1}t_\xi \text{ when } \xi > 0 \tag{2.14}$$

$$\frac{1}{2}\chi^2 - \chi\vartheta - \frac{\gamma^2}{(2k)} = \delta\beta^{-1}\left[\zeta + \int_\xi^0 (\mu\psi_{yy})_{y=\vartheta} d\xi\right], \quad \vartheta = O(\varepsilon) \text{ when } \xi < 0 \tag{2.15}$$

$$\beta = \delta\varepsilon^{-1}\Delta^{1+2/\alpha}Q_0^{-1/\alpha}, \quad \vartheta(+0) = 0 \tag{2.16}$$

where the following notation for the integral forces has been used

$$\zeta = \int_\vartheta^\chi (2t_2 + j_\xi)dy, \quad j = \int_y^\chi t_1 dy, \quad t = j|_{y=\vartheta}$$

In these variables, the slope of the bed in the transition zone and the inflow of mass across the surface as a consequence of the accumulation of sediment are estimated as $O(\varepsilon)$, and the slope of the upper surface as $O(\delta)$, which follows from the second equation of (2.14), where for glaciers, $\delta = 0.1$.

In this formulation, where the equations for the stream function and the profile of the upper surface of the ice hold in every domain of the ice flow, the smoothness of the above-mentioned functions is assumed. The matching condition for the lower surface along the joining line (the second equation of (2.16)) will therefore be the equation for determining the magnitude of the parameter β. The values of the parameter β and the sea level γ in the inner coordinates determine X_g and $H(0)$.

As a consequence of changing to dimensionless quantities in the transition zone, we have

$$h = O(1), \quad \varepsilon \to 0, \quad \delta \to 0$$

Asymptotic analysis of the first equation of (2.14) shows that $\beta = O(1)$ when $\varepsilon \to 0$ or else a sequence ε_n, $n \to \infty$ exists such that $t_\xi \to 0$ or $t_\xi \to \infty$ when $n \to \infty$. This contradicts the conclusion concerning the equality of the spatial scales of the transition zone and is equivalent to the requirement that the derivatives in the inner variables should be of the same order of magnitude when $\varepsilon_n \to 0$.

A similar analysis shows that

$$\beta = O(\delta^m), \quad \delta \to 0, \quad 0 \leqslant m \leqslant 1$$

Suppose $m > 0$. Then, after expanding the functions of problem (2.13)–(2.16) in powers of δ, we obtain that the leading terms of the expansion are found neglecting the effect of the hydrostatic force (the first term in the first equation of (2.14)), while this force determines the dynamics of the shelf. This points to the fact that, when $m > 0$, it is impossible to match the outer expansion for the shelf and the inner expansion for the transition zone. Finally, we have

$$\beta = O(1), \quad \varepsilon \to 0, \quad \delta \to 0$$

Putting $\beta = 1$ in the first equation of (2.6), we find the characteristic scale for the ice thickness along the joining line

$$[z_t] = \{\eta[f_s][x](\rho_i g\delta)^{-\alpha}\}^{1/(\alpha+2)}$$

Equating $[z_t] = [z_s]$ in (2.11), the characteristic magnitude of the slope of the surface of the shelf close to the joining line $\varepsilon_t = 2^{-2\alpha-1}$ can be found. This quantity is small when $\alpha \geqslant 2$.

We will seek a solution of system (2.13)–(2.16) in the form of asymptotic expansions in powers of ε and δ

$$v = (\psi, \vartheta, h, \beta) = v_0 + O(\varepsilon) + O(\delta), \quad d = (\chi, \gamma) = d_0 + \delta d_1 + O(\varepsilon) + O(\delta^2), \quad k = 1 - \delta$$

For the terms of the inner expansion, we have

$$t_{1yy} + 2t_{2\xi y} - t_{1\xi\xi} = 0, \quad -\infty < \xi < +\infty, \quad 1 - h_0 < y < 1$$
$$y = 1: \quad \psi_{0yy} = 0, \quad \psi_0 = 1; \quad y = 0, \quad \xi < 0: \quad \psi_0 = \psi_{0y} = 0 \tag{2.17}$$
$$y = 1 - h_0, \quad \xi > 0: \quad e_1(1 - h_{0\xi}^2) = -2h_{0\xi}e_2, \quad \psi_0 = 0; \quad \beta_0 = \delta\varepsilon^{-1}\Delta^{1+2/a}Q_0^{-1/\alpha}$$

where

$$e_1 = \psi_{0yy} - \psi_{0\xi\xi}, \quad e_2 = 2\psi_{0\xi y}, \quad (t_1, t_2) = \mu(e_1, e_2), \quad \mu = (e_1^2 + e_2^2)^{(1-\alpha)/(2\alpha)}$$

The equation for the ice thickness has the form

$$\frac{1}{2}\beta_0 h_0^2 + h_0 t_\xi = \int_{1-h_0}^{1} \zeta dy \text{ when } \xi > 0; \; h_0(0) = 1 \tag{2.18}$$

The terms of the expansion for the lower surface (ϑ) and the upper surface (χ) of the glacier and the sea level γ are given by the explicit formulae

$$\vartheta_0 = 0 \text{ when } \xi < 0; \quad \vartheta_0 = 1 - h_0 \text{ when } \xi > 0; \quad \gamma_0 = 1, \quad \gamma_1 = -1 - \beta_0^{-1} t_\xi\big|_{\xi=0}$$
$$\chi_0 = 1, \forall\xi; \quad \chi_1 = h_0 - 1 + \beta_0^{-1}(t_\xi - t_\xi\big|_{\xi=0}) \text{ when } \xi > 0 \tag{2.19}$$
$$\chi_1 = \beta_0^{-1}\left[\int_0^1 \zeta dy + \int_\xi^0 (\mu\psi_{0yy})_{y=0} d\xi - t_\xi\big|_{\xi=0}\right] - \frac{1}{2} \text{ when } \xi < 0$$

where

$$j = \int_y^1 t_1 dy, \quad t = j\big|_{y=1-h_0}, \quad \zeta = 2t_2 + j_\xi$$

Equation (2.17) for the stream function is defined in an infinite strip with a plane horizontal upper boundary and an unknown lower boundary when $\xi > 0$. When $\xi < 0$, that is, in the domain of the land-based glacier, the lower boundary is also a horizontal straight line. The first terms of the expansion for the height of the upper surface and the sea level are found using formulae (2.19).

The solution of problem (2.17)–(2.19) can be obtained by numerical methods since an analytical solution is difficult. It is remarkable that, in these variables, the determination of the functions ψ, h and ϑ with an accuracy $O(\delta)$ enables one to find γ and χ with an accuracy of $O(\delta^2)$. The parameters determining the effect of the dynamics of the land-based part of the glacier and the shelf do not occur in the system of equations obtained and, therefore, the magnitude of β depends solely on the rheology of the ice.

Uniformly valid expansion. We will now describe an algorithm for constructing a uniformly valid solution for the profiles of the ice surfaces. Suppose problem (2.17)–(2.19) has been solved and the inner expansion of the functions ψ, ϑ, γ, χ, β has been determined in the transition zone. A knowledge of β and γ enables one to determine the position of the joining line X_g from the first relation of (2.16), that determines the parameters β_0

$$\varepsilon Q_0^{1/\alpha}\beta_0 = \delta[(L - B)/(\gamma_0 + \delta\gamma_1)]^{1+2/\alpha} \tag{2.20}$$

which is written in the variables of the land-based glacier, where $Q_0(X)$, L, $B(X)$ are known functions.

If solution (2.20) does not exist, then a steady ice flow also does not exists under the given conditions.

After the position of the joining line has been determined, the ice thickness along it is found

$$\Delta = (L - B)(\gamma_0 + \delta\gamma_1)^{-1}$$

On taking account of the fact that

$$\psi_0 = \overline{\psi} + O(\xi^{-2}), \quad \xi \to -\infty, \quad \overline{\psi} = [(\alpha+2)y - 1 + (1-y)^{\alpha+2}](\alpha+1)^{-1}$$

and matching the inner and outer expansions of the profile of the upper surface of the ice in the land-based part of the glacier, we obtain the uniformly valid expansion (in inner variables)

$$\chi = \overline{S}_0 / \Delta + 1 + \delta\chi_1 - \chi^* \quad \text{when} \quad \xi < 0; \quad \chi^* = 1 + \delta[\lambda - (\alpha+2)^{1/\alpha}\beta_0^{-1}\xi]$$

$$\lambda = \beta_0^{-1}\left[\int_{-\infty}^{0} (\mu\psi_{0yy} - \overline{\psi}_{yy}^{1/\alpha}\Big|_{\xi=0})_{y=0} d\xi - t_\xi\Big|_{\xi=0}\right] - \frac{1}{2}$$

The quantity χ_1 is determined using formulae (2.19), and $\overline{S}_0$ is the solution of Eq. (2.9) with the boundary condition on the joining line

$$\overline{S}_0(0) = \Delta(1 + \delta\lambda) + O(\delta^2)$$

In order to match the expansions which have been constructed in the floating ice domain, it is necessary to expand the functions of the outer expansion for the shelf in asymptotic series in δ, as was done in the case of the transition zone. An outer expansion of the inner expansion for the ice thickness can be obtained from formula (2.12), neglecting the accumulation of ice on the surface. Finally, for the ice thickness and the height of the ice surfaces, we obtain the uniformly valid expansions (in inner variables)

$$h = \tilde{H}_0\Delta_s / \Delta + h_0 - h^*, \quad \chi = \tilde{S}_0\Delta_s / \Delta + 1 + \delta\chi_1 - \chi^*, \quad \vartheta = \chi - h \quad \text{when} \quad \xi > 0$$

where

$$h^* = [2^{2\alpha+1}(\alpha+1)^{-1}\beta_0^{-\alpha} / (A_1 + \xi)]^{1/(\alpha+1)}, \quad \chi^* = 1 + \delta(\gamma_1 + h^*)$$

$$A = A_1\varepsilon\Delta(\alpha+1)Q_0^\alpha$$

The function h_0 and the constant A_1 are determined by the solution of problem (2.17), (2.18) and $\tilde{H}_0$ (when the constant A is known), and χ_1, γ_1 are determined using formula (2.12) and (2.19).

Numerical solution for the transition zone problem. System (2.17)–(2.19) was solved numerically for ice with constant viscosity. In this case, the domain of definition, an infinite strip with an unknown lower boundary, was truncated far from the joining line ($\xi_l < \xi < \xi_r$) and was then reduced to a rectangle by the change of variables $\zeta = (y - 1 + h_0)/h_0$. In these new variables, the equation for the ice thickness (2.18), when the stream function is known, will be a third-order equation. The additional boundary conditions

$$h_{0\xi}(0) = h_{0\xi\xi}(0) = 0, \quad h_0(\xi_r) = h_r$$

the last of which, as the calculations showed, only has a weak effect on the solution, were used to solve it.

The conditions along the joining line which have been presented were obtained from the following reasoning. If $h_0 \equiv 1$, then the asymptotic solution of the problem for stream function (2.17) when $(\xi, y) \to (0, 0)$ can be found [16]. If this solution is substituted into the equation for the lower surface of the ice, that has been reduced to a form which only includes the values of the stream function and its derivatives in the lowest surface [9], then the function $h_0 \equiv 1$ satisfies this equation. This enables us to assume that the function of the lower surface is continuous along the joining line.

We obtain

$$\beta_0 = 1.5, \quad \gamma_1 = 0.67, \quad \lambda = 0.32, \quad A_1 = 1.86$$

from the results of the calculations.

The calculated outer surfaces of the ice (line *a*) close to the joining line are shown in Fig. 2. The solutions for the land-based part of the glacier and the shelf, found in the thin-layer approximation, are shown by curves *c*. The sea level is shown by line *b*. It can be seen that the thin-layer approximation is accurate outside the neighbourhood of the joining line in some thicknesses. The existence of a local minimum of the function for the upper surface of the glacier close to the joining line is characteristic.

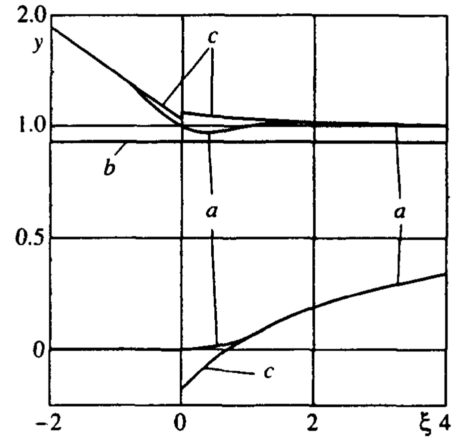

Fig. 2

Results of calculations of the dimensionless vertical velocity ($w = -\psi_{0\xi}$) and the horizontal velocity ($u = \psi_{0y}$), as well as the shear stress ($t_1 = \psi_{0yy} - \psi_{0\xi\xi}$) and the normal deviator stress ($t_2 = 2\psi_{0\xi y}$) are shown in Figs 3 and 4. Each curve corresponds to the distribution of the function over a cross-section and the number on a curve represents the distance of this cross-section from the joining line ξ in the thicknesses of the ice on it. For clarity, a vertical coordinate $\zeta = (y - 1 + h_0)/h_0$ is used in which the lower and the upper surfaces are defined as $\zeta = 0$ and $\zeta = 1$ respectively. For example, the label −1/2 on the left-hand side of Fig. 3 shows a graph of the distribution of the vertical velocity through the thickness of the ice at a distance from the joining line towards the land-based glacier equal to 1/2 of the ice thickness along the joining line. At a distance of one thickness of the ice (the label is −1), the flow is close to Poiseuille flow, the vertical velocity is close to zero and the horizontal velocity has an exponential profile (the right-hand side of Fig. 3). At a distance of two thicknesses of the ice towards the shelf (label 2), the flow is close to expansion flow (piston flow) when the horizontal velocity is constant throughout the depth (the right-hand side of Fig. 3) which, as a consequence of the mass balance, leads to a linear profile of the vertical velocity. Since the inflow of mass through the surface is neglected in the transition zone, the surface of the ice is a material surface and as a consequence of this, the vertical velocity is equal to zero on the upper (horizontal) surface and depends on the horizontal velocity and the slope of the given surface in the lower (inclined) surface. It is clear that the nature of the flow changes sharply in a domain, the length of which is equal to three thicknesses of the ice. Outside this domain, the thin-layer approximation is acceptable.

Conclusion. The analysis carried out in Section 2 enabled us to find a rigorous boundary condition on the joining line (2.20). The parameter β is determined solely by the solution of the inner problem, although it can also be found from natural measurements in the case of natural glaciers.

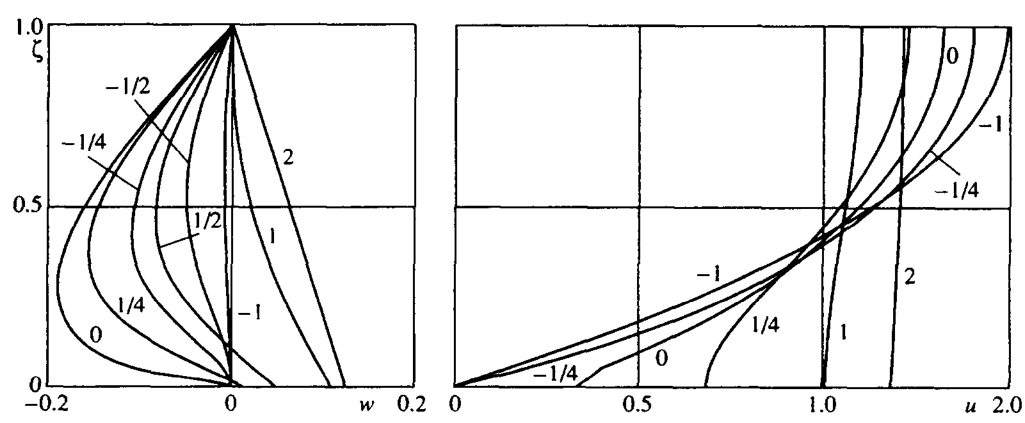

Fig. 3

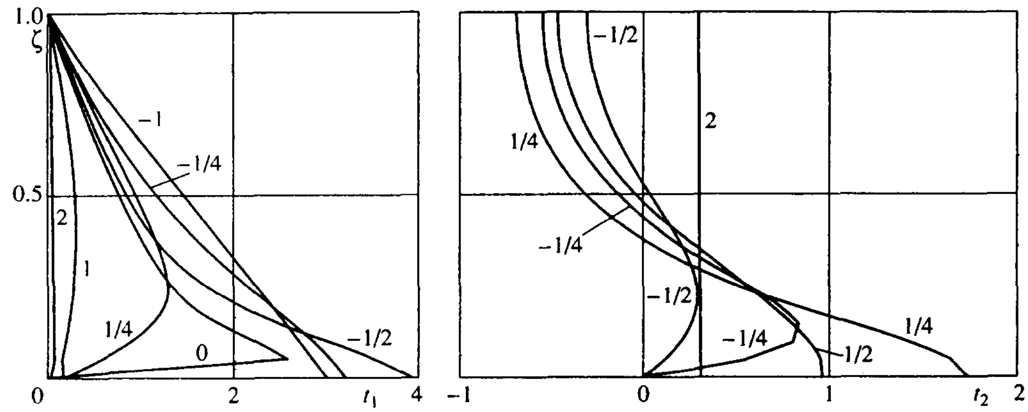

Fig. 4

Numerical solution of the problem for the transition zone determined the existence of a local minimum in the profile of the upper surface of the glacier close to the joining line which can be used to observe it in the case of field investigations of glaciers which are interacting with an ocean.

If follows from the above analysis that, along the joining line, the ice is close to a state of hydrostatic equilibrium (in the general case, the error is estimated as $O(\delta) = 10\%$). In the case of a constant viscosity of the ice, numerical calculations showed that the ice thickness corresponding to the inner solution for the transition domain and the outer solution for the land-based glacier along the joining line differs from the ice thickness when the ice is at hydrostatic equilibrium by –33.% and 0.1% respectively. The condition of hydrostatic equilibrium of ice in water can therefore be used as a boundary condition for the land-based part of a glacier the along joining line.

The complex approach to the study of a whole glaciological system (the land-based part, the shelf and the transition zone) enables one to establish a link between the characteristic scales of its parts in terms of the parameters ε and δ, which characterize the external conditions of existence of the glacier and, also, to construct a well-founded simplified closed model which describes its dynamics and takes account of the special features of all of its zones.

3. ANALYSIS OF THE DYNAMICS OF THE ICE-DIVIDE DOMAIN IN THE CASE OF AXISYMMETRIC ICE FLOW

Next, we will consider an unsteady, axisymmetric flow of ice where, when analysing a particular zone of the ice divide, it is sufficient to consider the dynamics of just the land-based glacier (in this case, a glacier which is completely located on a land can be considered). Knowledge of the position of the end of the glacier X_g and the height of the surface on it Δ (where $\Delta = 0$ for a land-based glacier) is sufficient in order to formulate the problem for the leading terms of the expansion in closed form since Eq. (3.5) (see below) for determining the leading term of the outer expansion of the function for the profile of the upper ice surface is a first-order integrodifferential equation in a spatial variable.

After changing to dimensionless variables, problem (1.1)–(1.5), as in the modelling of plane flow, can be reduced to the problem of determining the stream function $\Psi(U = \Psi_Z/R,\ W = -\Psi_R/R)$ and the unknown ice surface S when $n = 1$ (new variables are denoted by capital letters, where $R = x/[x]$).

$$T_{1ZZ} = \varepsilon^2[(V(RT_1)_R)_R + T_{3RZ} + VT_{4Z} - V(RT_{2Z})_R], \quad 0 < R < R_g, \quad B < Z < S \tag{3.1}$$

$$Z = S:\ \ \varepsilon^2 S_R(T_2 - T_3) = (1 - \varepsilon^2 S_R^2)T_1, \quad \Psi = \int_0^R R(F_s - S_\Theta)dR \tag{3.2}$$

$$Z = B:\ \ \Psi = \Psi_Z = 0; \quad R = 0:\ \ \Psi_Z = (V\Psi_R)_R = 0; \quad S(R_g) = \Delta \tag{3.3}$$

where

$$E_1 = V\Psi_{ZZ} - \varepsilon^2 (V\Psi_R)_R, \quad E_2 = (V\Psi_Z)_R, \quad E_3 = -V\Psi_{RZ}, \quad E_4 = V^2\Psi_Z, \quad V = 1/R$$
$$\mu = [E_1^2 + 2\varepsilon^2 (E_2^2 + E_3^2 + E_4^2)]^{(1-\alpha)/(2\alpha)} \quad (T_1, T_2, T_3, T_4) = \mu(E_1, 2E_2, 2E_3, 2E_4)$$

The equation for the profile of the upper ice surface has the form

$$V\Sigma_R = B_R[H + \varepsilon^2 I|_{Z=B}] + (1 + \varepsilon^2 B_R^2)[\mu V\Psi_{ZZ}]_{Z=B} + V\Phi \tag{3.4}$$

where

$$V\Sigma = -\frac{H^2}{2} + \varepsilon^2 \int_B^S (-I + T_2)dZ, \quad \Phi = -\frac{H^2}{2} + \varepsilon^2 \int_B^S (-I + T_4)dZ$$
$$I = T_3 - V\left(\int_Z^S RT_1 dZ\right)_R, \quad H = S - B$$

Equation (3.4) was obtained by integrating the first equation of (2.1) with respect to z from b to s and is the equation for the balance of the forces in a vertical cross-section x = const.

Model of a land-based glacier. We will seek the outer expansions $\overline{\Psi}$, $\overline{S}$ for the stream function Ψ and the function for the height of the surface S far from the ice divide in the form

$$\overline{J} = \overline{J}_0 + O(\varepsilon^2), \quad J = (\Psi, S)$$

Then, from relations (3.1)–(3.4), we find, in a similar manner to that described previously in [17].

$$\overline{\Psi}_0 = \frac{RQ}{\alpha+1}\left[(\alpha+2)\frac{Z-B}{\overline{H}_0} - 1 + \frac{(\overline{S}_0 - Z)^{\alpha+2}}{\overline{H}_0^{\alpha+2}}\right] \tag{3.5}$$
$$\overline{S}_{0R} = -\left[(\alpha+2)\frac{Q}{\overline{H}_0^{\alpha+2}}\right]^{1/\alpha}, \quad \overline{S}_0(R_g) = \Delta$$

where

$$\overline{H}_0 = \overline{S}_0 - B, \quad Q = V\int_0^R R(F_s - \overline{S}_{0\Theta})dR$$

The equation for the profile of the upper ice surface (the second equation of (3.5)) is a first-order integro-differential equation. It can be reduced to a differential equation of parabolic type with the additional boundary condition $\overline{S}_{0R}(\Theta, 0) = 0$. It can be solved numerically since it is difficult to find an analytical solution in the case of an arbitrary form of the functions B and F_s. If the bed is plane and horizontal and, also, the rate of accumulation is constant, a steady solution can be found analytically [17].

Close to the ice divide ($R \to 0$), we obtain

$$\overline{\Psi}_0 = \Omega R(\alpha+1)^{-1}\left[(\alpha+2)\frac{Z}{\overline{S}_0} - 1 + \frac{(\overline{S}_0 - Z)^{\alpha+2}}{\overline{S}_0^{\alpha+2}}\right]_{R=0} + O(R^{1+1/\alpha}) \tag{3.6}$$
$$\overline{S}_0 = \overline{S}_0(0) - \overline{S}_0(0)^{-1-2/\alpha}\alpha(\alpha+1)^{-1}(\alpha+2)^{1/\alpha}\Omega^{1/\alpha}R^{1+1/\alpha} + O(R^{2+1/\alpha})$$
$$\Omega = \frac{1}{2}(F_s - \overline{S}_{0\Theta})_{R=0}$$

The second equation of (3.5) serves to determine the profile of the glacier surface far from the ice divide. It follows from formula (3.1) that an outer expansion which makes use of the smallness of the ratio of the normal deviator stress to the shear stress is unsuitable when $R = O(\varepsilon)$ since, in this case, $\overline{\Psi}_{ZZ} = O(\varepsilon^2)$ and the shear and normal stresses are of the same order of magnitude. Hence, in the ice-divide domain, it is necessary to find an internal expansion. In spite of the fact that the second equation of (3.5) is unsuitable close to the ice divide, this solution is necessary in order to find the ice thickness on it, which is determined solely using the matching procedure. It is clear that, as a consequence of neglecting the rate of tensile deformations compared with the rate of shear deformations in Eq. (3.1), the magnitudes of the stresses (and, consequently, of the velocities also) obtained using an outer expansion in the ice-divide domain differ from the true values by an amount which is comparable with the stresses (velocities) themselves. Moreover, as can be seen from the second equation of (3.6), the outer expansion of the function for the profile of the upper ice surface possesses an infinite curvature when $R = 0$.

Model of the ice-divide dynamics. The unknown thickness of the ice on the axis of symmetry of the glacier in the variables (R, Z), which will be determined from the matching, is denoted by $\Delta_d = S(0)$. We will now introduce the inner expansion variables in the ice-divide domain.

$$\xi = \frac{R}{\varepsilon\Delta_d}, \quad (y, \chi, \vartheta) = \frac{1}{\Delta_d}(Z, S, B), \quad \psi = \frac{2\Psi}{\varepsilon^2\Delta_d^2(F_s(0) - \Delta_{d\Theta})}$$

In the new variables, the equations for determining the stream function look similar to (3.1)–(3.3) if the symbols $(\varepsilon, R, Z, S, B, \Psi, T_k)$ are replaced by $(1, \xi, y, \chi, \vartheta, \psi, t_k)$ in these equations, where

$$\psi\big|_{y=\chi} = \xi + O(\varepsilon), \quad \vartheta = O(\varepsilon^2)$$

where the second equation follows from the smoothness and symmetry of the function for the bed profile when $R = 0$: $B_R(0) = 0$. Here, the equation of the profile of the upper surface (for the balance of forces in the cross-section) takes the form

$$-(\chi^2/2)_\xi = \varepsilon^{1+1/\alpha}\Omega_d\left\{\upsilon(\mu\psi_{yy})_{y=\vartheta} + \upsilon\int_\vartheta^\chi (t_4 - t_2)dy - \left[\int_\vartheta^\chi (-i + t_2)dy\right]_\xi\right\} + O(\varepsilon^2), \quad \chi(0) = 1$$

where

$$i = t_3 - \upsilon\left(\int_y^\chi \xi t_1 dy\right)_\xi, \quad \upsilon = \frac{1}{\xi}, \quad \Omega_d = \Delta_d^{-1-1/\alpha}\left(\frac{F_s(0) - \Delta_{d\Theta}}{2}\right)^{1/\alpha}$$

We will seek the inner expansion in the form

$$\psi = \psi_0 + O(\varepsilon), \quad \vartheta = 0 + O(\varepsilon^2), \quad \chi = \chi_0 + \varepsilon^{1+1/\alpha}\chi_1 + O(\varepsilon^2)$$

For the terms of the expansion, we have the system

$$\begin{aligned} &t_{1yy} + \upsilon(\xi t_{2y})_\xi - (\upsilon(\xi t_1)_\xi)_\xi - t_{3\xi y} = \upsilon t_{4y}, \quad 0 < \xi < +\infty, \quad 0 < y < 1 \\ &y = 1:\ e_1 = 0, \ \psi_0 = \xi; \quad y = 0:\ \psi_0 = \psi_{0y} = 0 \\ &\xi = 0:\ \psi_{0y} = (\upsilon\psi_{0\xi})_\xi = 0 \end{aligned} \tag{3.7}$$

The terms of the expansion for the height of the surface are determined using the explicit formulae

$$\chi_0 = 1, \quad \chi_1 = \Omega_d\left\{\int_0^1 (-i + t_2)dy\Bigg|_0^\xi - \int_0^\xi \upsilon\left[(\mu\psi_{yy})_{y=0} + \int_0^1 (t_4 - t_2)dy\right]d\xi\right\}$$

where

$$e_1 = \upsilon\psi_{0yy} - (\upsilon\psi_{0\xi})_\xi, \quad e_2 = (\upsilon\psi_{0y})_\xi, \quad e_3 = -\upsilon\psi_{0\xi y}, \quad e_4 = \upsilon^2\psi_{0y}, \quad \upsilon = 1/\xi$$

$$\mu = [e_1^2 + 2(e_2^2 + e_3^2 + e_4^2)]^{(1-\alpha)/(2\alpha)}, \quad (t_1, t_2, t_3, t_4) = \mu(e_1, 2e_2, 2e_3, 2e_4)$$

The resulting system of equations describes the flow of the glacier in the ice-divide domain. Note that the first term in the expansion for the height of the upper surface is completely determined by the zeroth term of the expansion for the stream function. It is easily verified that, in the case of a Newtonian fluid ($\alpha = 1$), the outer expansion (the thin-layer approximation) (3.6) for the stream function satisfies the inner problem (3.7) with an accuracy $O(\varepsilon^2)$.

Problem (3.7) for the normalized stream function ψ_0 does not include parameters which depend on time, unlike the first formula of (3.6) for the stream function $\bar{\Psi}_0$.

The slope of the upper surface of the glacier can be neglected with an error $O(\varepsilon^{1+1/\alpha})$ since the domain of definition of problem (3.7) is an infinite layer of unit thickness with horizontal, plane boundaries. However, the uniformly valid expansion for the upper surface, found with an accuracy of $O(\varepsilon^{1+1/\alpha})$ (that is, after matching the functions χ_0 and $\bar{S}_0$), will have an infinite curvature when $\xi = 0$ and, therefore, when matching, it makes sense to take account of the next term in the expansion for χ, that is, χ_1. The quantity Δ_d is determined by the condition for matching the inner expansion with the outer expansion.

On taking account of relation (3.6) and the fact that

$$\psi_0 = \psi^\infty + O(1), \quad \xi \to \infty; \quad \psi^\infty = \frac{\xi^2}{\alpha+1}[(\alpha+2)y - 1 + (1-y)^{\alpha+2}]$$

after matching, we obtain the uniformly valid solution for the profile of the glacier surface (in inner variables)

$$\chi = 1 + \varepsilon^{1+1/\alpha}\chi_1 + \frac{\bar{S}_0}{\Delta_d} - \chi^*$$

where

$$\chi_1 = \Omega_0\left\{\zeta|_0^\xi - \int_0^\varepsilon \upsilon[(\mu\psi_{0yy})_{y=0} + \omega]d\xi\right\}, \quad \zeta = \int_0^1 (-i + t_2)dy$$

$$\Delta_d = \bar{S}_0(0)(1 - \varepsilon^{1+1/\alpha}\lambda), \quad \Omega_0 = \bar{S}_0(0)^{-1-1/\alpha}(F_s/2 - \bar{S}_{0\Theta}/2)^{1/\alpha}_{\xi=0}$$

$$\chi^* = 1 + \varepsilon^{1+1/\alpha}\Omega_0[C - \alpha(\alpha+2)^{1/\alpha}(\alpha+1)^{-1}\xi^{1+1/\alpha}]$$

$$C = -\int_0^{+\infty} \upsilon\{[\mu\psi_{0yy} - (\psi^\infty_{yy})^{1/\alpha}]_{y=0} + \omega\}d\xi - \zeta\big|_{\xi=0}, \quad \lambda = \Omega_0 C, \quad \omega = \int_0^1 (t_4 - t_2)dy$$

If follows from this solution and the second formula of (3.6) that the graph of the function $\bar{S}_0/\Delta - \chi^*$ has a finite curvature on the axis of symmetry of the glacier. On the other hand, smooth functions occur in the expression for χ_1 (it is assumed that the stream function, a solution of system (3.7), has continuous fourth-order derivatives). The solution which has been constructed is therefore characterized by a finite curvature of the upper surface on the axis of symmetry of the glacier.

Conclusion. The investigations carried out in Section 3 have shown that, in the case of axisymmetric ice flow in the ice-divide domain, all the unknown characteristics can be calculated in an infinite layer with plane horizontal boundaries, the thickness of which is identical, to a sufficient degree of accuracy, with that found using the thin-layer (shear flow) simplification.

The problem, describing the dynamics of the ice-divide domain, can be reduced, by a change of variables, to a form which does not include parameters which depend on time, the length of the glacier, the rate of accumulation of ice on the surface, the rheological parameter and the ice thickness, and it can be solved separately from the problem for the glacier as a whole.

The solution of the steady problem for the ice-divide domain enables one to determine its unsteady fields by an extension of the coordinate axes and renormalization of the stream function.

The curvature of the upper surface of the glacier is finite along its axis of symmetry.

The thin-layer simplification cannot be used to calculate the velocities and stresses in the ice-divide domain.

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